Quantitative Literacy:
Thinking Between the Lines

Crauder, Evans, Johnson, Noell

Chapter 9: Geometry

Chapter 9 Geometry
Lesson Plan

- Perimeter, area, and volume: How do I measure?
- Proportionality and similarity: Changing the scale
- Symmetries and tilings: Form and patterns

Learning Objectives:

- Calculate perimeters, areas, volumes, and surface areas of familiar figures.
  - Finding the area: Reminders about circles, rectangles, and triangles
  - Applications of basic geometry formulas
  - Heron's formula
  - Three-dimensional objects

9.1 Perimeter, area, and volume: How do I measure?

- **Circle:** a figure where all points are a fixed distance, the *radius*, from a fixed point, the *center*.
- For a circle of radius r:

$$\text{Area} = \pi r^2$$
$$\text{Circumference} = 2\pi r$$

The number $\pi \approx 3.14159$.

The formula for the circumference of a circle can be rewritten as:

$$\frac{\text{Circumference}}{\text{Diamter}} = \pi$$

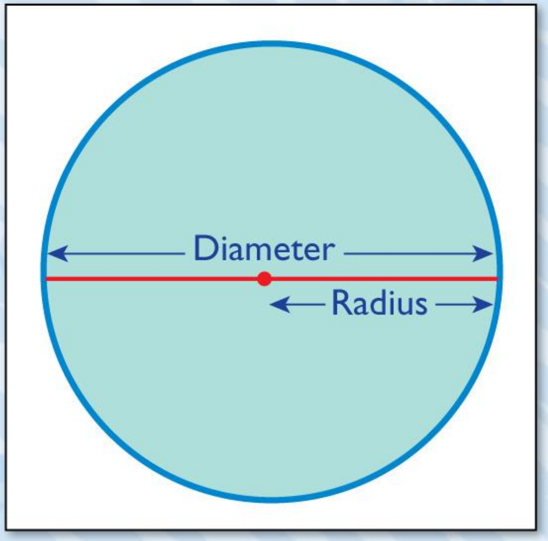

FIGURE 9.2 Radius and diameter of a circle.

This tells us that the ratio of circumference to diameter is always the same no matter which circle you study. The ratio is the same for the equator of earth and the equator of a baseball.

9.1 Perimeter, area, and volume: How do I measure?

- **Rectangles:** A rectangle has four right angles, and opposite sides are equal. Here are two basic formulas:

$$\text{Area} = \text{Length} \times \text{Width} \qquad \text{Perimeter} = 2 \times \text{Length} + 2 \times \text{Width}$$

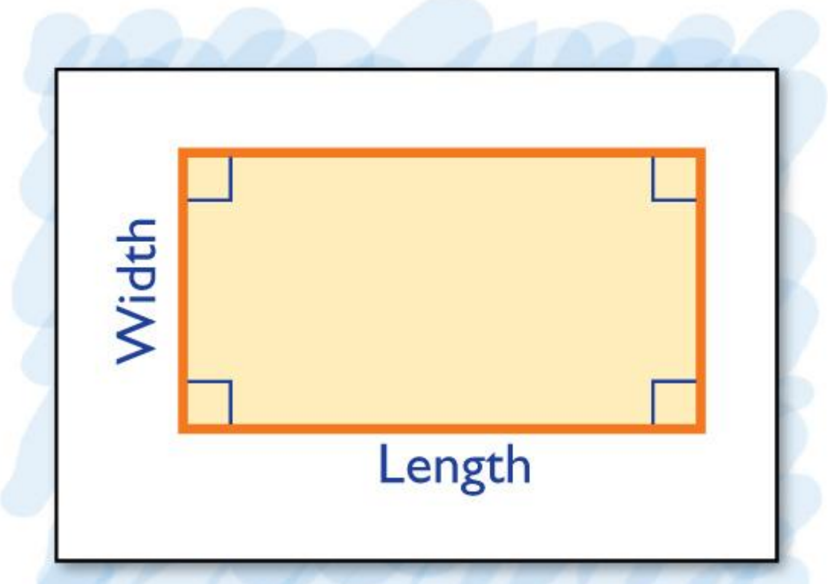

FIGURE 9.3 Opposite sides of a rectangle are of equal length.

9.1 Perimeter, area, and volume: How do I measure?

- **Triangles:** A triangle is a figure with three sides.

  To find the area of a triangle, we select any one of the three sides and label it the *base*. Then we find the *height* by starting at the vertex opposite the base and drawing a line segment that meets the base in a right angle.

- The **perimeter** of a geometric figure is the distance around it.
  1. A circle, the perimeter is **circumference** and equals 2π times the radius.
  2. A rectangle, the perimeter is the sum of the lengths of its four sides.
  3. A triangle, the perimeter is the sum of the lengths of its three sides.

- The **area** of a geometric figure measures the region enclosed by the figure.
  1. A circle, the area is π times the radius squared.
  2. A rectangle, the area is the product of the length and the width.
  3. A triangle, the area is one-half the product of the length of the base times the length of the height.

Chapter 9 Geometry

9.1 Perimeter, area, and volume: How do I measure?

- In Figure 9.4, we have chosen the base to be *AB*, and the resulting height is "inside" the triangle.
- In Figure 9.5, we have chosen the base to be the side *AC*, which results in a height "outside" the triangle. No matter which side we choose for the base, the area is always given by the formula

$$\text{Area} = \frac{1}{2}\text{Base} \times \text{Height}$$

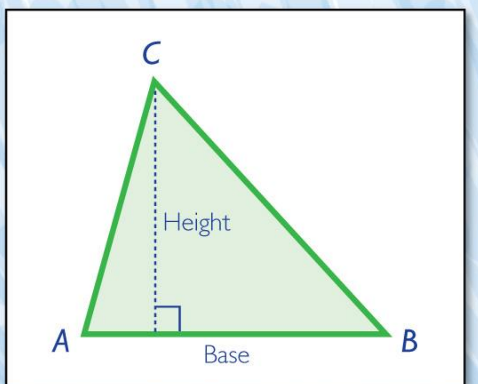

FIGURE 9.4 Base with height inside the triangle.

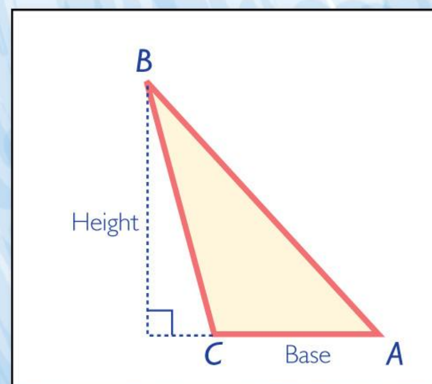

FIGURE 9.5 Different base with height outside the triangle.

9.1 Perimeter, area, and volume: How do I measure?

- **Right triangles:** A *right triangle* is a triangle with one 90-degree (or right) angle. One of the most familiar of all facts in geometry is the famous theorem of Pythagoras.

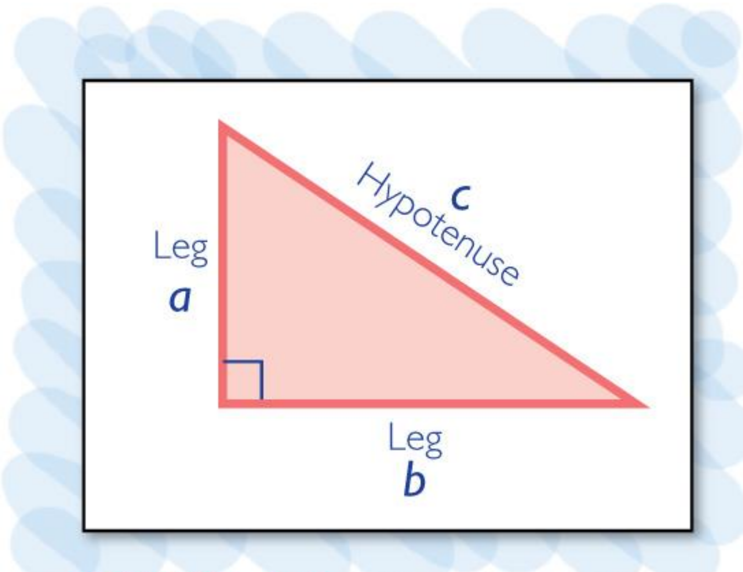

FIGURE 9.6 The sides of a right triangle.

- **Pythagorean theorem** for a right triangle:

$$a^2 + b^2 = c^2$$

In this formula, c always represents the *hypotenuse* (the side opposite the right angle). The other two sides of a right triangle are the *legs*.

The converse of the Pythagorean theorem is true: If a triangle has three sides of lengths $a, b,$ and c such that $a^2 + b^2 = c^2$, then that triangle is a right triangle with hypotenuse c.

9.1 Perimeter, area, and volume: How do I measure?

- **Example:** Suppose two runners A and B are side-by-side, 2 feet apart, on a circular track, as seen in Figure 9.7. If they both run one lap, staying in their lanes, how much farther did the outside runner B go than the inside runner A? (Note that no information was given about the diameter of the track.)

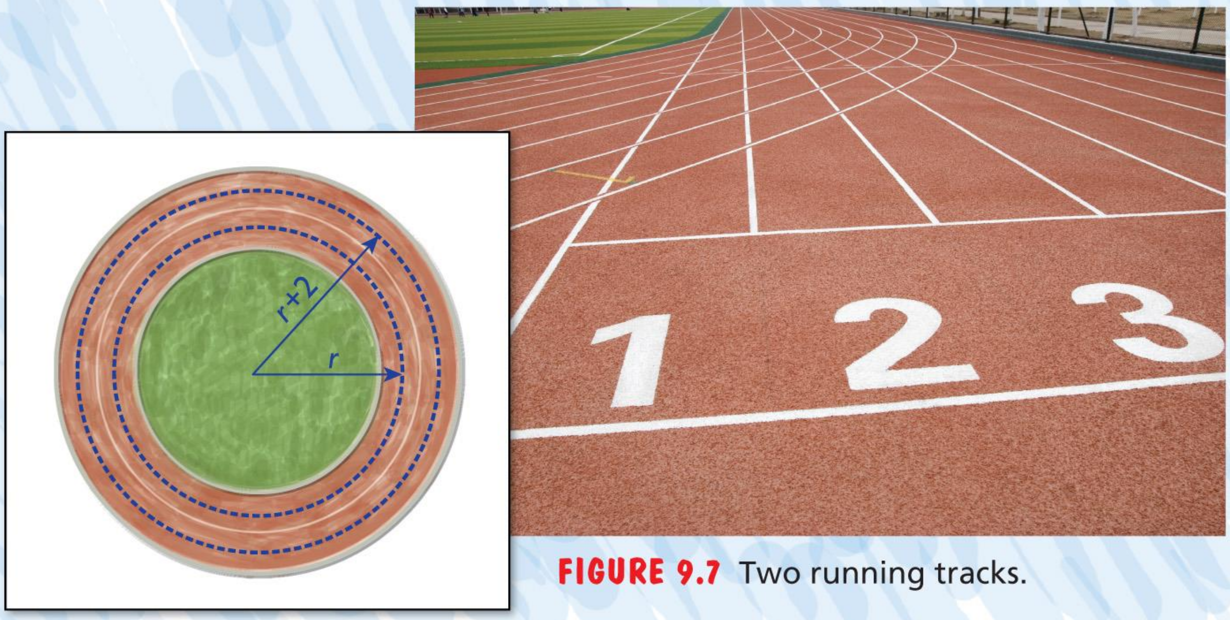

FIGURE 9.7 Two running tracks.

- **Solution:**

Let r denote the radius of the inside lane, where A is running.

Then $r + 2$ is the radius of the outside lane, where B is running.

The distance covered by runner A:

$$\text{Length of inside track} = 2\pi r$$

The distanced covered by B:

$$\text{Length of outside track} = 2\pi(r + 2)$$
$$= 2\pi r + (2\pi \times 2) = \text{Length of inside track} + 4\pi$$

Therefore, the distances covered by the two runners differ by 4π or about 12.6 feet. It may appear that a longer inside track would cause a greater difference in the distance the runners travel.

But the difference is the same, 12.6 feet, whether the inside track is 100 yards in diameter or 100 miles in diameter.

9.1 Perimeter, area, and volume: How do I measure?

- **Example:** At a local Italian restaurant a 16-inch-diameter pizza costs \$15 and a 12-inch-diameter pizza costs \$10.

  Which one is the better value?

- **Solution:** The area of each pizza in terms of its radius *r*:

$$\text{Area} = \pi r^2$$

The 16-inch-diameter pizza: $r = 8$ inches

$$\text{Area of 16-in.-diameter pizza} = \pi \times (8\text{ in.})^2 = 64\pi\text{ in.}^2$$

The 12-inch-diameter pizza: $r = 6$ inches

$$\text{Area of 12-in.-diameter pizza} = \pi \times (6\text{ in.})^2 = 36\pi\text{ in.}^2$$

The larger pizza costs: $\$15/64\pi$ square inches or $\$0.075$ per in^2.

The smaller pizza costs: $\$10/36\pi$ square inches or 0.088 per in^2.

Thus, the larger pizza is the better value.

- **Example:** Hockey player A is located 5 vertical yards and 7 horizontal yards from the goal, and hockey player B is 8 vertical yards and 3 horizontal yards from the goal. (See Figure 9.8)

  Which player would have the longer shot at the goal?

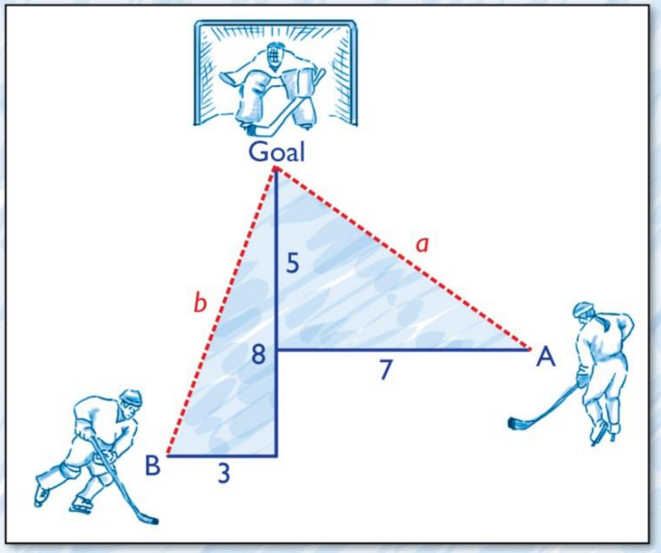

FIGURE 9.8 Two hockey players.

- **Solution:** The distance for player A is marked *a* in the figure.

  It represents the hypotenuse of a right triangle.

  Similarly, *b* is the distance for player B, and it represents the hypotenuse of a right triangle.

  We use the Pythagorean theorem:

$$a^2 = 5^2 + 7^2 = 72$$
$$a = \sqrt{74} \approx 8.6 \text{ yards}$$

$$b^2 = 8^2 + 3^2 = 73$$
$$b = \sqrt{73} \approx 8.5 \text{ yards}$$

  Player A has the longer shot.

- **Heron's formula:** gives the area of a triangle if we know the lengths of all three sides. (See Figure 9.10 with sides *a, b,* and *c)*

$$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$$

where *S is the semi-perimeter* (1/2 of the perimeter) of the triangle:

$$S = \frac{1}{2}(a + b + c)$$

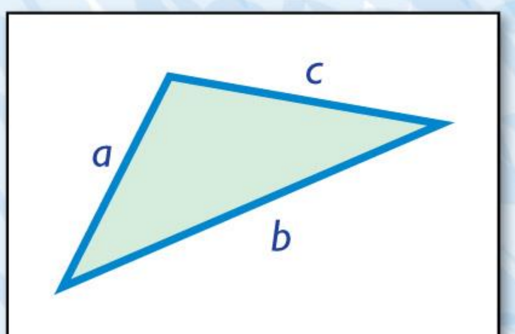

FIGURE 9.10 A triangle with given sides.

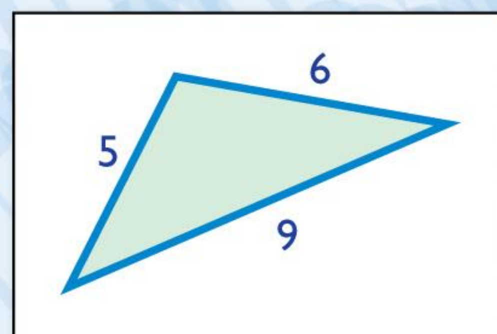

FIGURE 9.11 Applying Heron's formula.

- **Example:** For the triangle in Figure 9.11, which has sides 5, 6, and 9, find the area of a triangle.
- **Solution:** We find that:

$$S = \frac{1}{2}(5 + 6 + 9) = 10$$

The area is:

$$\text{Area} = \sqrt{10(10-5)(10-6)(10-9)} = \sqrt{200}$$

$$\approx 14.1 \text{ square units.}$$

- **Example:** My lawn has the shape shown in Figure 9.12.

1. I want to fence my lawn this spring. How many feet of fencing will be required?
2. I want to fertilize my lawn with bags of fertilizer that cover 500 square feet each. How many bags of fertilizer do I need to buy?

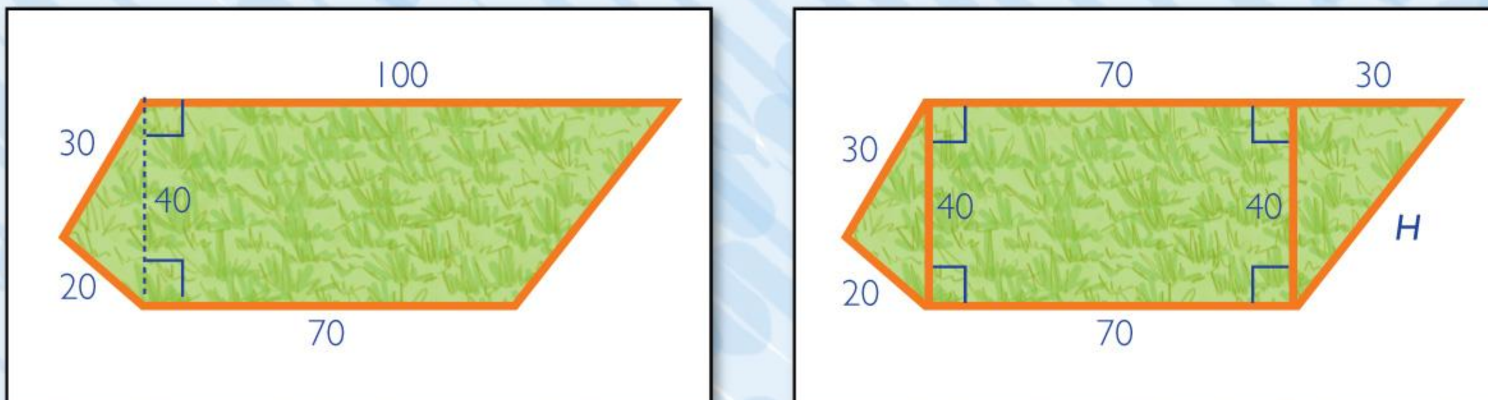

FIGURE 9.12 My lawn.

FIGURE 9.13 Dividing into a rectangle and two triangles.

- **Solution:**

1. The lawn has an irregular shape, by adding the line segment shown in Figure 9.13, we can divide it into three pieces: a right triangle, a rectangle, and an oblique triangle.

To find the perimeter, we need to find the hypotenuse (marked *H* in Figure 9.13) of the right triangle. The legs have length 30 feet, 40 feet. We find *H* using the Pythagorean theorem:

$$H^2 = 30^2 + 40^2$$
$$H^2 = 2500$$
$$H = 50 \text{ feet}$$

Now we calculate the perimeter of the lawn as:

$$\text{Perimeter} = 30 + 20 + 70 + 50 + 30 + 70 = 270 \text{ feet}$$

- **Solution (cont.):**

2. The area of the lawn is the area of the rectangle plus the area of the right triangle plus the area of the oblique triangle.

$$\text{Area of rectangle} = \text{Length} \times \text{Width} = 70 \times 40 = 2800 \text{ square feet}$$

$$\text{Area of right triangle} = \frac{1}{2}\text{ Base} \times \text{Height} = \frac{1}{2} \times 30 \times 40 = 600 \text{ square feet}$$

Heron's formula to calculate the area of the oblique triangle. The semi - perimeter is $S = \frac{1}{2}(40 + 30 + 20) = 45$:

$$\text{Area of oblique triangle} = \sqrt{45(45 - 40)(45 - 30)(45 - 20)}$$

$$= \sqrt{84{,}375} \approx 290 \text{ square feet}$$

Thus, the total area of the lawn is $2800 + 600 + 290 = 3690$ square feet. Each bag of fertilizer covers 500 square feet. Dividing 3690 by 500 gives about 7.4. Therefore, we will need to buy eight bags of fertilizer.

9.1 Perimeter, area, and volume: How do I measure?

- Labeling the dimensions of a box as *length*, *width*, and *height* as shown in Figure 9.14:

$$\text{Volume of a box} = \text{Length} \times \text{Width} \times \text{Height}$$

For example, the volume of the box in Figure 9.14 is:

$2 \text{ units} \times 3 \text{ units} \times 5 \text{ units} = 30 \text{ cubic units}$

Note that the volume of a box can also be expressed as the area of the base times the height:

$$\text{Volume} = \text{Area of base} \times \text{Height}$$

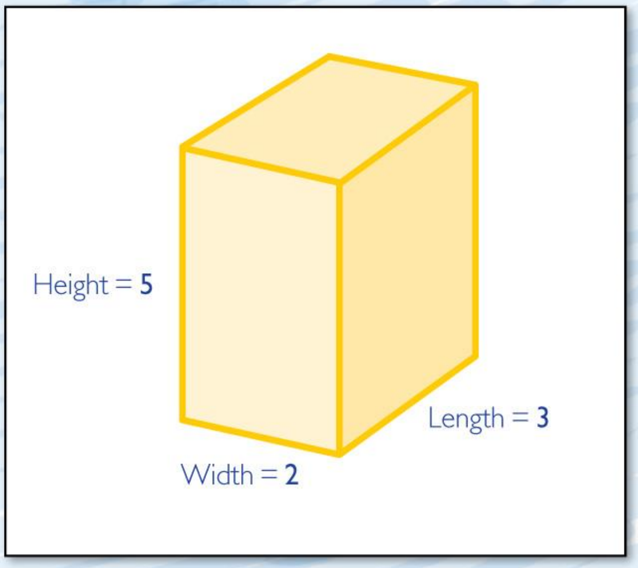

FIGURE 9.14 Dimensions of a box.

$$\text{Volume} = \text{Area of base} \times \text{Height}$$

This formula also holds for any three-dimensional object with uniform cross sections, such as a cylinder.

A cylinder has uniform circular cross sections, and a box has uniform rectangular cross sections.

For the cylinder shown in Figure 9.15, the radius is *r*, so the area of the base is πr^2. Using h for the height,

$$\text{Volume} = \text{Area of base} \times \text{Height}$$

$$= \pi r^2 h$$

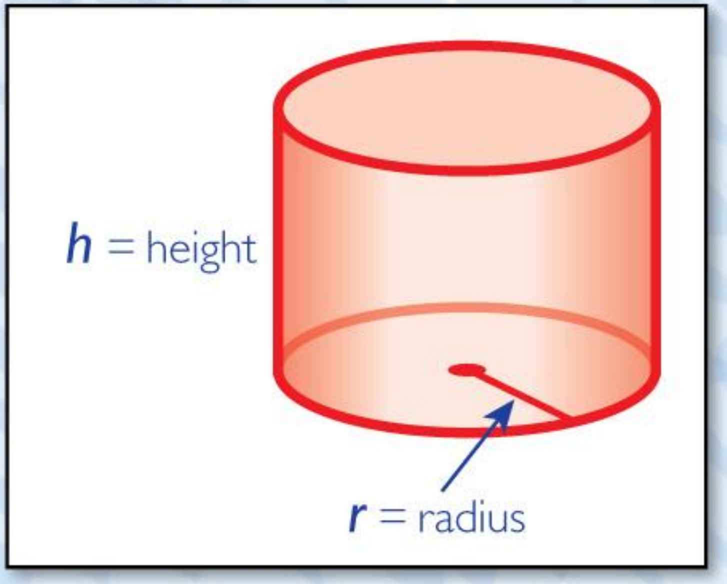

FIGURE 9.15 A cylinder.

9.1 Perimeter, area, and volume: How do I measure?

- **Example:** What is the volume of a cylindrical wading pool that is 6 feet across and 15 inches high?
- **Solution:** Because the pool is 6 feet across, the diameter of the circular base is 6 feet. The diameter is given in feet and the height is given in inches, so we need to convert units for one of them.

  Let's change the diameter of 6 feet to 72 inches.

  Then the radius $r = 36$ inches, and the height $h = 15$ inches.

$$\begin{aligned} \text{Volume} &= \text{Area of base} \times \text{Height} \\ &= \pi r^2 h \\ &= \pi(36 \text{ inches})^2 \times 15 \text{ inches} \end{aligned}$$

  This is about 61,073 cubic inches.

- **Example:** Cake batter rises when baked. The best results for baking cakes occur when the batter fills the pan to no more than two-thirds of the height of the pan. Suppose we have 7 cups of batter, which is about 101 cubic inches. We have a pan that is 2 inches high and has a square bottom that is 9 inches by 9 inches. Is this pan large enough?

- **Solution:** This pan forms the shape of a box, and we want to find the volume based on two-thirds of the full height 2 inches. Thus we use a height of:

$$\text{Height} = \frac{2}{3} \times 2 = \frac{4}{3}\,\text{inches}$$

We find:

$$\begin{aligned}\text{Volume} &= \text{Length} \times \text{Width} \times \text{Height} \\ &= 9\text{ inches} \times 9\text{ inches} \times \frac{4}{3}\,\text{inches} \\ &= 108\text{ cubic inches}\end{aligned}$$

This pan will easily hold batter with a volume of 101 cubic inches.

9.1 Perimeter, area, and volume: How do I measure?

- To find the **surface area** of a cylinder of radius *r* and height *h* (excluding the top and bottom), think of the cylinder as a can that we split lengthwise and roll out flat. (See Figure 9.16)

This gives a rectangle with width *h* and length equal to the circumference of the circular base.

The circumference of the base is $2\pi r$, so the surface area of the cylinder (excluding the top and bottom):

Surface area of cylinder $= 2\pi rh$

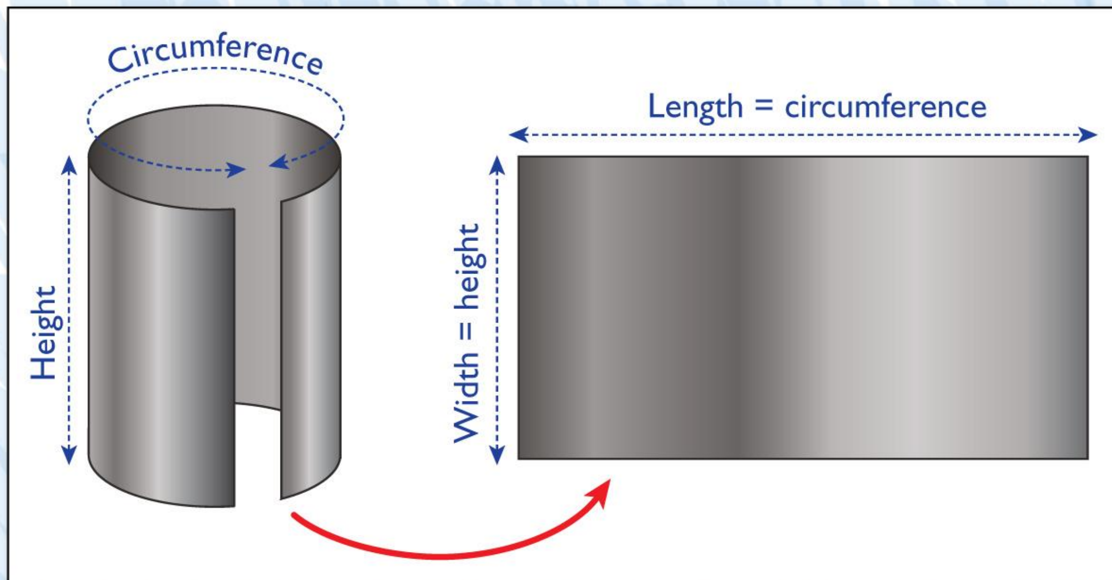

FIGURE 9.16 Splitting the cylindrical side of a can and rolling it out flat.

- **Example:** A tin can has a radius of 1 inch and a height of 6 inches.

1. How much liquid will the can hold?
2. How much metal is needed to make the can?

- **Solution:**

1. The base is a circle of radius 1:

$$\text{Area of base} = \pi \times 1^2 = \pi \text{ square inches}$$

The height is 6 inches, so the volume is:

$$\text{Volume} = \text{Area of base} \times \text{Height} = \pi \times 6 \text{ cubic inches}$$

This is about 18.8 cubic inches.

2. The metal needed to make the can consists of the top, bottom, and cylindrical side.

- **Solution (cont.):**

2. We already found that the base has area π square inches.

$$\text{Area of top and bottom} = 2\pi \text{ square inches}$$

To find the area of the cylindrical side, we use the formula $2\pi rh$ for the surface area of a cylinder:

$$\text{Area of side} = 2\pi \times 1 \times 6 = 12\pi \text{ square inches}$$

The total area includes the top and bottom of the can:

$$2\pi + 12\pi = 14\pi \approx 44.0 \text{ square inches}$$

Volumes and Surface Areas

- The volume of a box is:

$$\text{Volume of a box} = \text{Length} \times \text{Width} \times \text{Height}$$

- The volume of a cylinder is:

$$\text{Area of base} \times \text{Height}$$

If the cylinder has radius r and height h, this equals $\pi r^2 h$.

- The surface area of a cylinder (excluding the top and bottom) of radius r and height h is $2\pi r h$.

Chapter 9 Geometry: **Chapter Summary**

- **Perimeter, area, and volume:** How do I measure?
  - Geometric objects can be measured in terms of perimeter (circumference), area and volume.
  - The Pythagorean Theorem of a right triangle
- **Proportionality and similarity:** Changing the scale
  - Proportional and its constant of proportionality
  - Golden rectangles
  - Similar triangles
- **Symmetries and tiling:** Form and patterns
  - Rotational symmetry and reflectional symmetry about a line
  - Tiling or tessellation
  - Regular tiling